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ANALYTIC SOLUTIONS OF THE $(2+1)$-DIMENSIONAL NONLINEAR EVOLUTION EQUATIONS USING THE EXTENDED TANH METHOD

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Abstract

The extended tanh method is used to construct exact periodic and soliton solutions of $(2+1)$-dimensional nonlinear evolution equations. The compactons solutions, solitary wave solutions, solitary patterns solutions, and periodic wave solutions of the generalized $(2+1)$-dimensional Boussinesq, breaking soliton, and BKP equations are successfully obtained. These solutions may be important of significance for the explanation of some practical physical problems. It is shown that the extended tanh method provides a powerful mathematical tool for solving many great nonlinear partial differential equations in mathematical physics.

1. Introduction

Studies of various physical structures of nonlinear dispersive equations had attracted much attention in connection with the important

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problems that arise in scientific applications. Mathematically, these physical structures have been studied by using various analytical methods, such as inverse scattering method [1], Darboux transformation method [4, 9], Hirota bilinear method [5], Lie group method [2, 10], bifurcation method of dynamic systems [6, 11, 22], sine-cosine method [7, 8, 13, 15, 16, 18, 19], tanh function method [7, 8, 17, 20], Fan-expansion method [3, 21], homogenous balance method [14], and so on. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

Recently, by using the sine-cosine method, Tascana and Bekir [12] studied the following $(2+1)$-dimensional Boussinesq, breaking soliton, and BKP equations, respectively,

$$u_{tt} - u_{xx} - u_{yy} - (u^2)_{xx} - u_{xxxx=0}, \tag{1.1}$$

$$u_t + \alpha u_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0,\ u_y = v_x, \tag{1.2}$$

$$w_t = w_{xxx} + w_{yyy} + 6(uw)_x + 6(vw)_y,\ u_y = w_x,\ v_x = w_y. \tag{1.3}$$

It is shown that this class gives compactons, conventional solitons, solitary patterns, and periodic solutions.

In this paper, we will study the generalized forms of Equations (1.1), (1.2), and (1.3), which are written by

$$(u^n)_{tt} - (u^n)_{xx} - (u^n)_{yy} - (u^2)_{xx} - (u^n)_{xxxx=0}, \tag{1.4}$$

$$(u^n)_t + \alpha(u^n)_{xxy} + 4\alpha u v_x + 4\alpha u_x v = 0,\ u_y = v_x, \tag{1.5}$$

$$(w^n)_t = (w^n)_{xxx} + (w^n)_{yyy} + 6(uw)_x + 6(vw)_y,\ u_y = w_x,\ v_x = w_y, \tag{1.6}$$

where n is a non-zero integer, α is known constant.

It is the objective of this work to further complement in implementing the tanh method [7, 8] to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. The next interest is the determination of exact travelling wave solutions with distinct physical structures to the generalized

$(2+1)$-dimensional Boussinesq, breaking soliton, and BKP equations. Our approach depends mainly on the tanh method [7, 8] that has the advantage of reducing the nonlinear problem to a system of algebraic equations that can be solved by using Mapple or Mathematica. As stated before, our approach depends mainly on the extended tanh method. In what follows, we highlight the main steps of the proposed method.

2. Analysis of the Methods

For the extended tanh method, we first use the wave variable $\xi = x - ct$, to carry a PDE in two independent variables

$$P(u, u_t, u_x, u_{xx}, u_{xxx}, \dots) = 0, \tag{2.1}$$

into an ODE

$$Q(u, u', u'', u''', \dots) = 0. \tag{2.2}$$

Equation (2.2) is then integrated as long as all terms contain derivatives, where integration constants are considered zeros.

The standard tanh method introduced in [7, 8], where the tanh is used as a new variable, since all derivatives of a tanh are represented by a tanh itself. We use a new independent variable

$$Y = \tanh(\mu\xi), \tag{2.3}$$

that leads to the change of derivatives:

$$\frac{d}{d\xi} = \mu(1 - Y^2)\frac{d}{dY}, \quad \frac{d^2}{d\xi^2} = \mu^2(1 - Y^2)\left(-2Y\frac{d}{dY} + (1 - Y^2)\frac{d^2}{dY^2}\right). \tag{2.4}$$

We then apply the following finite expansion:

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k, \tag{2.5}$$

and

$$u(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}, \tag{2.6}$$

where M is a positive integer that will be determined to derive a closed form analytic solution. However, if M is not an integer, a transformation formula is usually used. Substituting (2.4) and (2.5) or (2.6) into the simplified ODE (2.2) results in an equation in powers of Y. To determine the parameter M, we usually balance the linear terms of highest order in the resulting equation with the highest order nonlinear terms. With M determined, we collect all coefficients of powers of Y in the resulting equation, where these coefficients have to vanish. This will give a system of algebraic equations involving the parameters $a_k (k = 0, \ldots, M)$, $b_k (k = 1, \ldots, M)$, μ, and c. Having determined these parameters, knowing that M is a positive integer in most cases, and using (2.5) or (2.6), we obtain an analytic solution $u(x, t)$ in a closed form.

3. Using the Extended Tanh Method

1. The generalized $(2+1)$-dimensional Boussinesq equation (1.4).

We begin first with the Equation (1.4). Using the wave variable $\xi = x + y - ct$, the Equation (1.4) is carried to ODE,

$$(c^2 - 2)(u^n)'' - (u^2)'' - (u^n)^{(4)} = 0. \tag{3.1}$$

Integrating (3.1) twice, respectively, using the constants of integration to be zero, we find

$$(c^2 - 2)u^n - u^2 - n[u^{n-1}u'' + (n-1)u^{n-2}(u')^2] = 0. \tag{3.2}$$

Using the assumptions of the tanh method, (2.4)-(2.6) gives

$$(c^2 - 2)S^n - S^2 - n(n-1)S^{n-2}\mu^2(1 - Y^2)^2\left(\frac{dS}{dY}\right)^2$$

$$- nS^{n-1}\mu^2(1 - Y^2)\left[\left(-2Y\frac{dS}{dY} + (1 - Y^2)\frac{d^2S}{dY^2}\right)\right] = 0. \tag{3.3}$$

To determine the parameter M, we usually balance S^2 in the resulting Equation (3.3) with the highest order nonlinear terms. This in turn gives

$$2M = M(n-1) + M + 2, \tag{3.4}$$

so that,

$$M = -\frac{2}{n-2}. \tag{3.5}$$

To get a closed form analytic solution, the parameter M should be an integer. A transformation formula

$$u = v^{-\frac{1}{n-2}}, \tag{3.6}$$

should be used to achieve our goal. This in turn transforms (3.2) to

$$(c^2 - 2)v^2 - v^3 - n\left[-\frac{1}{n-2}vv'' + \frac{2(n-1)}{(n-2)^2}(v')^2\right] = 0. \tag{3.7}$$

Balancing vv'' and v^3 gives $M = 2$. The extended tanh method allows us to use the substitution

$$v(x, t) = S(Y) = A_0 + A_1 Y + A_2 Y^2 + B_1 Y^{-1} + B_2 Y^{-2}. \tag{3.8}$$

Substituting (3.8) into (3.7), collecting the coefficients of each power of Y, and using Mapple to solve the resulting system of algebraic equations, we obtain the following three sets:

$$A_1 = B_1 = B_2 = 0,\ A_0 = A_0,\ A_2 = -A_0,\ \mu^2 = \frac{A_0(n-2)^2}{2n(n+2)},\ c^2 = -2 + \frac{2nA_0}{(n+2)}, \tag{3.9}$$

$$A_1 = B_1 = A_2 = 0,\ A_0 = A_0,\ B_2 = -A_0,\ \mu^2 = \frac{A_0(n-2)^2}{2n(n+2)},\ c^2 = -2 + \frac{2nA_0}{(n+2)}, \tag{3.10}$$

$$A_1 = B_1 = 0,\ A_0 = -2A_2 = -2B_2 = \frac{4n\mu^2(n+2)}{(n-2)^2},\ \mu^2 = \mu^2,\ c^2 = 2 - \frac{16n^2\mu^2}{(n+2)^2}. \tag{3.11}$$

Noting that $u = v^{-\frac{1}{n-2}}$, for $A_0 > \frac{n+2}{n}$, $n = 1, \pm 3, \pm 4, \cdots, \pm k, \cdots,$ or $A_0 < -1$, $n = -1$, we obtain the solitary wave solution and the solitary patterns solutions:

$$u(x, y, t) = \left[A_0 \left(1 - \tanh^2 \sqrt{\frac{A_0 (n-2)^2}{2n(n+2)}} \left(x + y \pm \sqrt{-2 + \frac{2nA_0}{n+2}} t \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.12}$$

$$u(x, y, t) = \left[A_0 \left(1 - \coth^2 \sqrt{\frac{A_0 (n-2)^2}{2n(n+2)}} \left(x + y \pm \sqrt{-2 + \frac{2nA_0}{n+2}} t \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.13}$$

$$u(x, y, t) = \left[A \left(1 - \frac{1}{2} \left(\tanh^2 \mu (x + y \pm ct) + \coth^2 \mu (x + y \pm ct) \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.14}$$

where $c = \sqrt{2 - \frac{16 n^2 \mu^2}{(n-2)^2}}$, $0 < \mu^2 < \frac{(n+2)^2}{8n^2}$, $A = \frac{4n\mu^2 (n+2)}{(n-2)^2}$.

However, for $A_0 < 0$, $n = 1, \pm 3, \pm 5, \cdots, \pm (2k+1), \cdots,$ or $A_0 > 0$, $n = -1$, we obtain the periodic and compactons solutions:

$$u(x, y, t) = \left[A_0 \left(1 + \tan^2 \sqrt{\frac{-A_0 (n-2)^2}{2n(n+2)}} \left(x + y \pm \sqrt{-2 + \frac{2nA_0}{n+2}} t \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.15}$$

$$u(x, y, t) = \left[A_0 \left(1 + \cot^2 \sqrt{\frac{-A_0 (n-2)^2}{2n(n+2)}} \left(x + y \pm \sqrt{-2 + \frac{2nA_0}{n+2}} t \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.16}$$

$$u(x, y, t) = \left[A \left(1 + \frac{1}{2} \left(\tan^2 \mu (x + y \pm ct) + \cot^2 \mu (x + y \pm ct) \right) \right) \right]^{-\frac{1}{n-2}}, \tag{3.17}$$

where $c = \sqrt{2 - \frac{16n^2\mu^2}{(n-2)^2}}, \quad \mu^2 < 0.$

2. The generalized $(2+1)$-dimensional breaking soliton equations (1.5).

We now consider the generalized $(2+1)$-dimensional breaking soliton equations (1.5). Using the wave variable $\xi = x + y - ct$, we find

$$-c(u^n)' + \alpha(u^n)''' + 4\alpha uv' + 4\alpha u'v = 0, \; u' = v'. \tag{3.18}$$

Integrating the first and the second equation in the system and neglecting constants of integration, we find

$$-cu^n + \alpha(u^n)'' + 4\alpha uv = 0, \; u = v. \tag{3.19}$$

Substituting the second of (3.19) into the first equation of (3.19), we find

$$-cu^n + \alpha(u^n)'' + 4\alpha u^2 = 0. \tag{3.20}$$

Utilizing the same procedure as before, we get a series of exact solutions for Equation (3.20), which include solitary patterns solutions, compactons solutions, and periodic solutions. Three of which are solitary wave solutions and the solitary patterns solutions:

$$u(x, y, t) = \left[\frac{1}{2}A_0\left(1 - \tanh^2\sqrt{\frac{-c(n-2)^2}{4\alpha n(3n-4)}}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.21}$$

$$u(x, y, t) = \left[\frac{1}{2}A_0\left(1 - \coth^2\sqrt{\frac{-c(n-2)^2}{4\alpha n(3n-4)}}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.22}$$

$$u(x, y, t) = \left[A_0\left(1 - \frac{1}{2}\left(\tanh^2\mu(x + y - ct) + \coth^2\mu(x + y - ct)\right)\right)\right]^{-\frac{1}{n-2}}, \tag{3.23}$$

where $A_0 = \dfrac{c(7n-10)}{16\alpha(3n-4)}$, $\mu = \pm\sqrt{\dfrac{-c(n-2)^2}{16\alpha(3n-4)}}$, $c\alpha < 0$, $n = -1, \pm 3, \cdots,$ $\pm(2k+1), \cdots,$ or $c\alpha > 0$, $n = 1$.

However, for $c\alpha > 0$, $n = -1, -2, \pm 3, \pm 4, \cdots, \pm k, \cdots,$ or $c\alpha < 0$, $n = 1$, we obtain the periodic and compactons solutions:

$$u(x, y, t) = \left[\frac{1}{2}A_0\left(1 + \tan^2\sqrt{\frac{c(n-2)^2}{4\alpha n(3n-4)}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.24}$$

$$u(x, y, t) = \left[\frac{1}{2}A_0\left(1 + \cot^2\sqrt{\frac{c(n-2)^2}{4\alpha n(3n-4)}}(x+y-ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.25}$$

$$u(x, y, t) = \left[A_0\left(1 + \frac{1}{2}\left(\tan^2\mu(x+y-ct) + \cot^2\mu(x+y-ct)\right)\right)\right]^{-\frac{1}{n-2}}, \tag{3.26}$$

where $\mu = \pm\sqrt{\dfrac{c(n-2)^2}{16\alpha(3n-4)}}$.

3. The generalized $(2+1)$-dimensional BKP equation (1.6).

We next consider the generalized $(2+1)$-dimensional BKP equation (1.6). Using the wave variable $\xi = x + y - ct$, we find

$$-c(w^n)' = 2(w^n)''' + 6(uw)' + 6(vw)', \; u' = w', \; v' = w', \tag{3.27}$$

Integrating the first, second, and the third equation in the system and neglecting constants of integration, we find

$$-cw^n = 2(w^n)'' + 6(uw) + 6(vw), \; u = w, \; v = w. \tag{3.28}$$

Substituting the second and the third equation of (3.28) into the first equation of (3.28), we find

$$-cw^n = 2(w^n)'' + 12w^2. \tag{3.29}$$

Utilizing the same procedure as before, we get a series of exact solutions for Equation (3.29), which include solitary patterns solutions, compactons solutions, and periodic solutions. Three of which are solitary wave solutions and the solitary patterns solutions:

$$u(x, y, t) = \left[2A_0\left(1 - \tanh^2 \frac{n-2}{4n}\sqrt{-2c}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.30}$$

$$u(x, y, t) = \left[2A_0\left(1 - \coth^2 \frac{n-2}{4n}\sqrt{-2c}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.31}$$

$$u(x, y, t) = \left[A_0\left(1 - \frac{1}{2}\left(\tanh^2 \mu(x + y - ct) + \coth^2 \mu(x + y - ct)\right)\right)\right]^{-\frac{1}{n-2}}, \tag{3.32}$$

where $A_0 = \frac{-c(n+2)}{48n}$, $\mu = \pm\frac{n-2}{8n}\sqrt{-2c}$, $c < 0$, $n = \pm 1, \pm 3, \pm 4, \cdots,$ $\pm k, \cdots$.

However, for $c > 0$, $n = +1, \pm 3, \pm 5, \cdots, \pm(2k+1), \cdots,$ we obtain the periodic and compactons solutions:

$$u(x, y, t) = \left[2A_0\left(1 + \tan^2 \frac{n-2}{4n}\sqrt{2c}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.33}$$

$$u(x, y, t) = \left[2A_0\left(1 + \cot^2 \frac{n-2}{4n}\sqrt{2c}(x + y - ct)\right)\right]^{-\frac{1}{n-2}}, \tag{3.34}$$

$$u(x, y, t) = \left[A_0\left(1 + \frac{1}{2}\left(\tan^2 \mu(x + y - ct) + \cot^2 \mu(x + y - ct)\right)\right)\right]^{-\frac{1}{n-2}}, \tag{3.35}$$

where $\mu = \pm\frac{n-2}{8n}\sqrt{2c}$.

4. Discussion

The tanh method was used to investigate the generalized $(2+1)$-dimensional Boussinesq, breaking soliton, and BKP equations. The study revealed compactons, solitary, and periodic wave solutions for all examined variants. The study emphasized the fact that, the tanh method is reliable in handling nonlinear problems. The obtained results clearly demonstrate the efficiency of the method used in this work. Moreover, the method is capable of greatly minimizing the size of computational work compared to other existing techniques. In addition, specific restriction is usually required in that the value of M must be an integer to get closed form analytic solutions, therefore, transformation formula is required to overcome this difficulty. The tanh method worked successfully in handling nonlinear dispersive equations. This emphasizes the fact that, the tanh method is applicable to a wide variety of nonlinear problems.

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